International Journal of Theoretical Physics, Vol. 35, No. 5, 1996

Weyl, Curvature, Ricci, and Metric Tensor Symmetries

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Received March 14, 1995

Weyl symmetries for some specific spherically symmetric static spacetimes are derived and compared with metric, Ricci, and curvature tensor symmetries.

In recent years much interest has been shown in understanding the general theory of relativity in terms of Ricci collineations (RCs). It is argued that RCs can add to our understanding of the subject when compared with metric conservation laws (Bokhari and Qadir, 1990, 1993; Bokhari, 1990), which are given by Killing vectors (KVs). In this spirit, a complete classification of RCs was obtained for spherically symmetric static spacetimes (Bokhari and Qadir, 1993), which were compared with KVs admitted by the corresponding spacetimes; interesting results were obtained (Bokhari and Qadir, 1993; Bokhari *et al.*, 1994).

In their work on the symmetries of different tensors, Katzin *et al.* and Golub argue that the conservation laws on curvature tensors, called curvature collineations (CCs), may also yield new insights into the general theory of relativity when they are different from KVs or RCs (Katzin *et al.*, 1969; Katzin and Levine, 1972; Golab, 1972). However, until recently no serious attempt has been made to obtain a complete classification of the CCs. The main reason that the CCs remain unaddressed is that even in spherically symmetric static spacetime one has to deal with a highly overestimated system of 54 PDEs in four unknown functions, i.e., the four CCs given by ξ^α ($\alpha = 0, \ldots, 3$), of all the spacetime coordinates. Keeping this problem in mind, we recently addressed this problem for some specific spherically symmetric

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0020-7748/96/0500-1013$09.50/0

static metrics and the CCs obtained for those metrics (Bokhari *et al.*, n.d.; Kashif, 1995). These CCs were then compared with KVs and RCs and interesting results obtained.

While RCs and CCs can give more information than KVs, (Bokhari and Qadir, 1990, 1993; Bokhari, 1990; Bokhari *et al.*, 1994, n.d.; Khashif, 1995), the symmetries of the Weyl tensor, called Weyl collineations (WCs), may also provide some new understanding of the subject which is not given by KVs, RCs, or CCs. The reason for this claim is that the Weyl tensor is a function of Riemann, Ricci, and metric tensors apart from the Ricci scalar and hence their symmetries when put together may give rise to information not given by KVs, RCs, or CCs. This is the question that we address in this paper. However, like CCs, since it is difficult to deal with WC equations generally, we derive WCs only for some specific spherically symmetric static metrics and discuss them in the light of KVs, RCs, and CCs for the corresponding metrics.

A vector ξ is called a WC if the Lie derivative of the Weyl tensor along that vector is zero. Mathematically it is characterized by

$$\underset{\xi}{\mathcal{L}} W = 0 \tag{1}$$

where $\underset{\xi}{\mathcal{L}}$ represents the Lie derivative along ξ of the Weyl tensor W. In a torsion-free space, in a coordinate basis, equation (1) becomes

$$W^{\alpha}_{bcd,f}\xi^{f} + W^{\alpha}_{bcf}\xi^{f}_{,d} + W^{\alpha}_{bfd}\xi^{f}_{,c} + W^{\alpha}_{fcd}\xi^{f}_{,b} - W^{f}_{bcd}\xi^{\alpha}_{,f} = 0 \tag{2}$$

For spherically symmetric static spacetimes, these are 54 independent WE equations which can be obtained directly from the CC equations by replacing R^{α}_{bcd} by W^{α}_{bcd} there (Bokhari *et al.*, n.d.; Kashif, 1995). Also, since we solve WCs only for specific spherically symmetric static metrics, e.g., Minkowski, de Sitter/anti-de Sitter, Einstein/anti-Einstein, Schwarzschild, and some Bertotti–Robinson-like metrics, the Weyl tensor is either zero or is explicitly related to the metric tensors or the Riemann tensors for the corresponding metrics. Thus without giving derivations of WCs we only quote results.

In Minkowski spacetime all the $W^{\alpha}_{bcd} = 0$ trivially. Thus the WC equations (2) are satisfied for any arbitrary values of ξ^{α}. Thus in Minkowski spacetime every ξ^{α} is a WC.

In de Sitter/anti-de Sitter spacetimes, the Weyl tensor is related to the Riemann curvature tensor by

$$W^{\alpha}_{bcd} = -4R^{\alpha}_{bcd} \tag{3}$$

Taking the Lie derivative on both sides of this equation and requiring that this Lie derivative is zero, it turns out that the symmetries of the Weyl tensor are identically the same as the symmetries of the Riemann tensor. Thus, the WCs in this case turn out to be the same as the ten CCs (which are also the same as KVs or RCs in these metrics).

In the Einstein (anti-Einstein) metric

$$W^{\alpha}_{bcd} = 0 \tag{4}$$

Here, it is easily noticed that (2) are again satisfied for any ξ^{α}. This therefore implies that there exist arbitrary many WCs for the Einstein/anti-Einstein metrics. This is, however, an interesting case and we will discuss it in more detail later.

In the Schwarzschild metric the Weyl tensor is again the same as the Riemann tensor and hence the WCs turn out to be the same as four KVs or CCs there.

There exist in literature three spherically symmetric static Bertotti–Robinson-like metrics (Qadir and Ziad, 1993; Ziad, 1990)

$$\begin{aligned} ds_{\mathrm{I}}^2 &= (B + r)^2\, dt^2 - dr^2 - a^2\, d\Omega^2 \\ ds_{\mathrm{II}}^2 &= \cos^2(c + \sqrt{\alpha} r)\, dt^2 - dr^2 - a^2\, d\Omega^2 \\ ds_{\mathrm{III}}^2 &= \cosh^2(c + \sqrt{\alpha} r)\, dt^2 - dr^2 - a^2\, d\Omega^2 \end{aligned} \tag{5}$$

In all three cases the Weyl tensor components are given by

$$\begin{aligned} W^0_{101} &= \frac{1}{6} g_{11} R, \qquad W^2_{323} = \frac{1}{6} g_{33} R \\ W^0_{i0i} &= \frac{-1}{12} g_{ii} R = W^1_{i1i} \qquad (i = 2, 3) \end{aligned} \tag{6}$$

where g_{ii} is the metric corresponding to first, second, or third Bertotti–Robinson metric and R is the corresponding Ricci scalar. In the first Bertotti–Robinson metric, only KVs and WCs are the same, whereas in the other two cases besides KVs and WCs, the RCs as well as CCs are also the same. We will discuss these cases in more detail later.

We have computed the WCs in order to be able to understand how the information about the symmetries implicit in the geometry and relevant for the matter-energy field (through the gravitational field equations) is distributed among the metric tensor, the Ricci tensor, the Riemann tensor, and the Weyl tensor for some specific static spherically symmetric metrics. We discuss these results in the light of Table I for KVs, RCs, CCs, and WCs of the corresponding spacetimes.

From Table I it can be noticed that in the Minkowski spacetime except for the ten KVs, all of the RCs, CCs, and WCs are arbitrary. This is a direct consequence of the fact that the Minkowski metric is flat and all components

Table I. KVs, RCs, CCs, and WCs

Metric	KVs	RCs	CCs	WCs
Minkowski	10	Arbitrary	Arbitrary	Arbitrary
De Sitter/anti-de Sitter	10	10	10	10
Einstein/anti-Einstein	7	$6 + \xi^0$ (arb. x^α)	$6 + \xi^0$ (arb. t)	Arbitrary
Bertotti–Robinson$_{\text{I}}$	6	$3 + \xi^0(x^\alpha)$, $\xi^1(x^\alpha)$	$3 + \xi^0(t, r)$, $\xi^1(t, r)$	6
Bertotti–Robinson$_{\text{II}}$	6	6	6	6
Bertotti–Robinson$_{\text{III}}$	6	6	6	6
Schwarzschild	4	Arbitrary	4	4

of the Ricci, Riemann, and Weyl tensors are zero there. Thus every vector ξ^α in this metric is an RC, CC, or WC.

In the de Sitter/anti-de Sitter metrics, the Weyl, curvature, Ricci, and metric tensors are all constantly proportional to each other. Thus the KVs, RCs, CCs, and WCs are the same in these metrics.

In the Schwarzschild metric, whereas the WCs, CCs, and KVs are the same, the RCs are arbitrary.

The flat Bertotti–Robinson metric, ds_{I}^2 of equation (5), admits six WCs, which are the same as the six KVs there. However, since in this metric the $R^0_{i0i} = 0 = R_{00} = R_{11}$, the RCs and CCs become indefinite in ξ^0 and ξ^1. Further, the zero and one component of the RCs are functions of all the spacetime coordinates, whereas the same components of the CCs depend on time and radial coordinate only. Thus the flat Bertotti–Robinson metric admits six KVs, three spatial and arbitrary temporal and radial (functions of spacetime coordinates) RCs, three spatial and arbitrary temporal and radial (functions of time and r coordinate only) CCs, and six WCs in all. Similarly, the remaining two Bertotti–Robinson metrics also do not provide any extra information in terms of WCs, because they are the same as the CCs, RCs, or the KVs there.

The most important of the cases where WCs do provide additional information is that of the Einstein/anti-Einstein metrics. Here there is a fundamental difference between KVs, RCs, CCs, and WCs. Whereas these metrics admit seven KVs, the RCs and the CCs do not remain definite in temporal components because R_{00} and R^0_{i0i} $(i = 1, \ldots, 3)$ are zero there. Another difference between the RCs and CCs is that whereas the temporal component of the RCs depends arbitrarily on all the spacetime coordinates, the same component of the CCs can depend only on the time coordinate. Further, in these metric since all the Weyl tensor components are zero, the WC equations are satisfied for any arbitrary WC. Thus these metrics, besides six KVs, admit six spatial and arbitrary temporal (depending on all spacetime coordinates) RCs, six spatial and arbitrary temporal (depending on time coordinate only) CCs, and totally arbitrary WCs. Thus whereas KVs in the Einstein metrics are definite and only the temporal RC or CC becomes arbitrary, every direction ξ^α ($\alpha = 0, \ldots, 3$) is a WC.

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Erratum: Weyl, Curvature, Ricci, and Metric Tensor Symmetries[1]

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We have found an error in one of the results in our paper. Our claim in Eq. (3) is not true. According to the correct version, all the Weyl tensor components in de Sitter/anti-de Sitter spacetimes are zero identically and therefore give arbitrary WCs and not 10 as claimed in our paper.